The electromagnetic field equations for moving media

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In this paper a formulation of the field equation for moving media is developed by the generalization of an axiomatic geometric formulation of the electromagnetism in vacuum (Ivezić T 2005 *Found. Phys. Lett.* **18** 401). First, the field equations with bivectors $F(x)$ and $\mathcal{M}(x)$ are presented and then these equations are written with vectors $E(x)$, $B(x)$, $P(x)$ and $M(x)$. The latter ones contain both the velocity vector u of a moving medium and the velocity vector v of the observers who measure E and B fields. They do not appear in the entire previous literature. All these equations are written in the standard basis and compared with Maxwell's equations with 3-vectors. In this approach the Ampèr-Maxwell law and Gauss's law are inseparably connected in one law and the same happens with Faraday's law and the law that expresses the absence of magnetic charge. It is shown that Maxwell's equations with 3-vectors and our field equations with 4D geometric quantities are not equivalent in the 4D spacetime.

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1. Introduction

The field equations for moving media in a relativistically covariant formulation were first presented by Minkowski [1]. In [2], it was exposed an axiomatic geometric formulation of the electromagnetism in vacuum with only one axiom: the field equation for the bivector field F. In this paper the formulation from [2] is generalized to the moving media. The paper is organized as follows.

In section 2, the basic field equation for moving media (5) is expressed in terms of the bivector $F = F(x)$ that represents the electromagnetic field and the generalized magnetization-polarization bivector $\mathcal{M} = \mathcal{M}(x)$.

In section 3, the decomposition of F, equation (10), is presented. F is decomposed into the electric field vector E, the magnetic field vector B and the velocity vector v of the observers who measure E and B fields (in the usual notation E, B, v, ... are called 4-vectors). Similarly, the decomposition of $\mathcal{M}$ into the polarization vector $P(x)$, the magnetization vector $M(x)$ and the bulk velocity vector u of the medium is given by equation (12). Inserting equations (10) and (12) into the field equation (5) we find the general form of the field equation for a magnetized and polarized moving medium expressed in terms of $E(x)$, $B(x)$, $P(x)$ and $M(x)$, equation (15), i.e. equations (16) and (17), which are named the field equations in the Ampèrian form. In the equation (15) (and

(16)) there are two different velocities u and v and, as I am aware, these field equations do not appear in the entire previous literature. They are important results that are obtained in this paper.

In section 4, in order to compare (15), i.e. (16) and (17), with usual formulations that deal with 3-vectors, all quantities in (16) and (17) are represented in the standard basis $\{\gamma_\mu\}$, which yields equations (20) and (21), respectively.

In section 5, it is presented a brief review of the existence of the fundamental difference between the usual transformations (UT) of the electric and magnetic fields as the 3-vectors and the Lorentz transformations (LT) of the four-dimensional (4D) geometric quantities that represent the electric and magnetic fields in the 4D spacetime.

In sections 6 and 7, the equation (20) is applied to the case when the observers are at rest in a stationary medium and the case when the observers are at rest in the laboratory frame, but material medium is moving, respectively. Both cases are compared with the usual formulation with the 3-vectors.

In section 8 the discussion of the results and the conclusions are presented.

2. The basic field equation for moving media in terms of F and $\mathcal{M}$

We shall deal with 4D geometric quantities, i.e. in the geometric algebra formalism. For the exposition of the geometric algebra see [3]. The generators of the spacetime algebra are four basis vectors $\{\gamma_\mu\}, \mu = 0...3$, satisfying $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = diag(+---)$. This basis, the standard basis, is a right-handed orthonormal frame of vectors in the Minkowski spacetime M^4 with γ_0 in the forward light cone, $\gamma_0^2 = 1$ and $\gamma_k^2 = -1$ ($k = 1, 2, 3$). The standard basis $\{\gamma_\mu\}$ corresponds to Einstein's system of coordinates in which the Einstein synchronization of distant clocks [4] and Cartesian space coordinates x^i are used in the chosen inertial frame of reference.

The field equation in vacuum in the geometric algebra formalism is:

$$\partial F = j/\varepsilon_0 c, \quad \partial \cdot F + \partial \wedge F = j/\varepsilon_0 c. \tag{1}$$

It is shown in [2] that the bivector $F = F(x)$, which represent the electromagnetic field, can be taken as the primary quantity for the whole electromagnetism and the field equation for F, equation (1), is the basic equation. As shown in [2], the bivector field F yields the complete description of the electromagnetic field and, in fact, there is no need to introduce either the field vectors or the potentials. For the given sources the Clifford algebra formalism enables one to find in a simple way the electromagnetic field F, see equations (7) and (8) in [2].

If j is the total current density then (1) *holds unchanged in moving medium as well.* The equation (1) can be separated into the field equation with sources and that one without sources as

$$\partial \cdot F = j/\varepsilon_0 c, \quad \partial \wedge F = 0. \tag{2}$$

Since j is a vector the trivector part is identically zero (it holds only in the absence of a magnetic charge).

In some cases the total current density vector j can be decomposed as

$$j = j^{(C)} + j^{(\mathcal{M})}, \tag{3}$$

where $j^{(C)}$ is the conduction current density of the *free* charges and $j^{(M)}$ is the magnetization-polarization current density of the *bound* charges

$$j^{(\mathcal{M})} = -c\partial\mathcal{M} = -c\partial \cdot \mathcal{M} \tag{4}$$

($\partial \wedge \mathcal{M} = 0$, since $j^{(\mathcal{M})}$ is a vector). $\mathcal{M}$ is the generalized magnetization-polarization bivector $\mathcal{M} = \mathcal{M}(x)$.

Then (1) (i.e., (2)) can be written as

$$\partial(\varepsilon_0 F + \mathcal{M}) = j^{(C)}/c; \quad \partial \cdot (\varepsilon_0 F + \mathcal{M}) = j^{(C)}/c,\ \partial \wedge F = 0. \tag{5}$$

The trivector part, i.e., the field equation without sources, remained unchanged, because it is not affected by the separation of the current density vector j into free and bound parts; that part does not contain j. The equations (5) are the primary equations for the electromagnetism in moving media. In most materials $\mathcal{M}$ is a function of the field F and this dependence is determined by the constitutive relations. (They will be considered in a separate paper.) In that case (5) are well-defined equations for F.

Instead of dealing with the axiomatic formulation of electromagnetism for moving media that uses only the *local* form of the field equation (5) one can construct the equivalent integral form simply replacing F by $F + \mathcal{M}/\varepsilon_0$ in equations (18), (21), (22) and also j by $j^{(C)}$ in (21), (22) in [2]. However, the integral form will not be investigated here.

Proceeding in the same way as in [2] one can derive from (5) the stress-energy vector $T(n)$ for a moving medium simply replacing F by $F + \mathcal{M}/\varepsilon_0$ in equations (26), (37-47) in [2]. For example, equations (26) from [2] become

$$T(n) = T(n(x), x) = -(\varepsilon_0/2)\left\langle (F + \mathcal{M}/\varepsilon_0) n (F + \mathcal{M}/\varepsilon_0)\right\rangle_1. \tag{6}$$

$T(n)$ is a vector-valued linear function on the tangent space at each spacetime point x describing the flow of energy-momentum through a hypersurface with unit normal vector $n = n(x)$. The expression for $T(n)$, $T(n) = Un + (1/c)S$, equation (41) from [2], will remain unchanged, but the energy density U and the Poynting vector S will change according to the described replacement. All this with $T(n)$ will not be discussed in this paper, but in a separate paper in which also Abraham-Minkowski controversy will be examined in a new way.

Another form of the field equation with sources from (2) is the 'source representation'

$$\partial \cdot \varepsilon_0 F = j^{(C)}/c - \partial \cdot \mathcal{M}, \tag{7}$$

according to which the sources of the fundamental electromagnetic field F are the true currents $j^{(C)}$ and the magnetization-polarization current density $\partial \cdot$

$\mathcal{M}$, i.e., the space-time changes of the generalized magnetization-polarization bivector $\mathcal{M}$.

In all previous formulations of the electromagnetism in media (at rest, or moving), starting with Minkowski (his f_{hk}) [1], the electromagnetic excitation tensor is introduced, see, e.g., a modern textbook on classical electromagnetism [5], or the papers [6-8] and the references therein, in the recent - Annalen der Physik, Special Topic Issue 9-10/2008: The Minkowski spacetime of special relativity - 100 years after its discovery. Here, in (5), $\mathcal{H}$ can be introduced as

$$\mathcal{H} = \varepsilon_0 F + \mathcal{M}. \tag{8}$$

However, it is worth noting that (8) is in some sense unsatisfactory, since physically different kind of entities are mixed in it; an electromagnetic field F and a matter field $\mathcal{M}$, i.e., the magnetization-polarization bivector. Moreover, as will be seen in the next section, in general, two different velocity vectors, v - the velocity of the observers and u - the velocity of the moving medium, enter into the decompositions of F and $\mathcal{M}$, the equations (10) and (12), respectively. This fact causes that *the usual decomposition of* $\mathcal{H}$ *into the electric and magnetic excitations, equation* (14), *is not possible in the general case but only in the case if* $u = v$, or if both decompositions (10) and (12) are made with the same velocity vector, either u or v. In that case $\mathcal{H}$ can be introduced in (5) and the usual form of the field equations in moving media is obtained

$$\partial \cdot \mathcal{H} = j^{(C)}/c, \quad \partial \wedge F = 0. \tag{9}$$

3. The basic field equation for moving media in terms of E, B and P, M

In this paper instead of using (8) and (9) we deal with (5) (or (7)) as the basic field equation. In that equation bivectors F and $\mathcal{M}$ can be decomposed. First, the decomposition of F is considered. It is known that any antisymmetric tensor of the second rank can be decomposed into two space-like vectors and the unit time-like vector. When applied to the bivector F, e.g., equation (13) in [2], this yields

$$F = E \wedge v/c + (IcB) \cdot v/c, \tag{10}$$

where the electric and magnetic fields are represented by vectors $E(x)$ and $B(x)$ and I is the unit pseudoscalar.

Minkowski, section 11.6 in [1], see also [9], was the first who introduced vectors (4-vectors in the usual notation) of the electric and magnetic fields and the velocity vector, Φ, Ψ and w, respectively, in his notation, and presented the decomposition of F, his equation (55), that corresponds to (10). Note that he considered that w, Φ and Ψ are 1×4 matrices and F is a 4×4 matrix. Thus

he worked with components of the geometric quantities taken in the standard basis $\{\gamma_\mu\}$.

There is no rest frame for the field F, that is, for E and B, and therefore the vector v in the decomposition (10) has to be interpreted as the velocity vector of the observers who measure E and B fields. Then $E(x)$ and $B(x)$ are defined with respect to v, i.e., with respect to the observer, as

$$E = F \cdot v/c, \quad B = -(1/c)I(F \wedge v/c). \tag{11}$$

It also holds that $E \cdot v = B \cdot v = 0$; both E and B are space-like vectors. It is visible from (11) that E and B depend not only on F but on v as well. The unit pseudoscalar I is defined algebraically without introducing any reference frame, as in section 1.2. in the second reference in [3]. We choose I in such a way that when I is represented in the $\{\gamma_\mu\}$ basis it becomes $I = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3$. With such choice for I, $\{\gamma_1, \gamma_2, \gamma_3\}$ form a right-handed orthonormal set, as usual for a 3D Cartesian frame. The LT (boosts) do not change the orientation for spacetime. (Here, in the whole paper, under the name LT we shall only consider - boosts.) The relations corresponding to (11) are also first reported by Minkowski, section 11.6 in [1], see also [9].

Similarly, the bivector $\mathcal{M}(x)$ can be decomposed into two vectors, the polarization vector $P(x)$ and the magnetization vector $M(x)$ and the unit time-like vector u/c

$$\mathcal{M} = P \wedge u/c + (MI) \cdot u/c^2. \tag{12}$$

There is the rest frame for a medium, i.e., for $\mathcal{M}$, or P and M, and therefore the vector u in the decomposition (12) may be identified with bulk velocity vector of the medium in spacetime. Integral curves of u define the averaged world-lines of identifiable constituents of the medium. Then, $P(x)$ and $M(x)$ are defined with respect to u as

$$P = \mathcal{M} \cdot u/c, \quad M = cI(\mathcal{M} \wedge u/c) \tag{13}$$

and it holds that $P \cdot u = M \cdot u = 0$. As in the case with F, it can be seen from (13) that P and M depend not only on $\mathcal{M}$ but on u as well.

Usually, only the velocity vector u of the moving medium is taken into account, or the case $u = v$ is considered, i.e., it is supposed that the observer frame is comoving with medium, or both decompositions (10) and (12) are made with the same velocity vector, either u or v.

Such assumptions enable the introduction of the electromagnetic excitation bivector $\mathcal{H}$, equation (8), and, by using (10) and (12), one finds the decomposition of $\mathcal{H}$ into the electric and magnetic excitations (other names of which are 'electric displacement' and 'magnetic field intensity')

$$\mathcal{H} = D \wedge u/c + (IH) \cdot u/c^2, \tag{14}$$

where, as usual, the electric displacement vector $D = \varepsilon_0 E + P$ and the magnetic field intensity vector $H = (1/\mu_0)B - M$ are introduced. The decomposition (14) was first introduced by Minkowski, equation (56) section 11.6 in [1]. Notice that

Minkowski dealt only with bulk velocity vector of the medium u; in both his equations (55) (our equation (10) but with $v = u$) and (56) (our equation (14)) the vector w (our u) appears. The same treatment with the decomposition of $\mathcal{H}$ and consequently with only one velocity, the velocity u, is used in the latter textbooks, e.g., [10], and papers, e.g., [11]. However, in general, $u \neq v$, e.g., the observers are at rest in the laboratory frame ($v = c\gamma_0$) in which the considered medium is moving with velocity u ($u \neq c\gamma_0$). Therefore, we continue with an alternative approach which deals with two different velocity vectors v and u, i.e., with equations (10) and (12).

Inserting equations (10) and (12) into the field equation (5) one gets the general form of the field equation for a magnetized and polarized moving medium expressed in terms of $E(x)$, $B(x)$, $P(x)$ and $M(x)$

$$\partial\{\varepsilon_0[E \wedge v/c + (IB) \cdot v] + [P \wedge u/c + (1/c^2)(MI) \cdot u]\} = j^{(C)}/c. \qquad (15)$$

In the same way as in (5), the equation (15) with the geometric product can be divided into the vector part (with sources)

$$\partial \cdot \{\varepsilon_0[E \wedge v/c + (IB) \cdot v] + [P \wedge u/c + (1/c^2)(MI) \cdot u]\} = j^{(C)}/c \qquad (16)$$

and the trivector part (without sources)

$$\partial \wedge [E \wedge v/c + (IB) \cdot v] = 0. \qquad (17)$$

The field equation without sources (17) remained unchanged relative to the corresponding equation for vacuum, because the same holds for the trivector part of (5). We call equation (15), i.e. equations (16) and (17), the field equations in the Ampèrian form, in analogy with Maxwell's equations when they are written in terms of the 3-vectors **E**, **B**, **P** and **M**; for the latter ones and the name see, e.g. equations (4.5) in [12]. Observe that in equations (15), i.e., (16), there are two different velocities u and v. The equation (15) is a fundamental result, which is not previously reported in the physics literature, as I am aware.

If the geometric product is used then there is *only one equation for the electromagnetism in moving media, the equation* (5), i.e., in the Ampèrian form equation (15). They are written with abstract 4D geometric quantities and *they comprise and generalize all usual Maxwell's equations (with 3-vectors) for moving media.*

The equation (16) can be written in another form, i.e. in the 'source representation' as with F and $\mathcal{M}$, equation (7),

$$\partial \cdot \{\varepsilon_0[E \wedge v/c + (IB) \cdot v]\} = j^{(C)}/c - \partial \cdot [P \wedge u/c + (1/c^2)(MI) \cdot u], \qquad (18)$$

according to which the sources of E and B fields are $j^{(C)}$ and P and M. In that form it is clear that *it is not possible to separate the field equation with sources for the E field from that one for the B field.* Thus, *the usual Ampèr-Maxwell law and Gauss's law are inseparably connected in one law - equation* (16), i.e. *equation* (18). Similarly, *in equation* (17), *Faraday's law and the law that expresses the absence of magnetic charge are also inseparably connected in*

one law. This is an essential difference relative to Maxwell's equations with the 3-vectors **E**, **B**, **P** and **M**. Of course, the same statement holds for the original equation (2), i.e., for the vacuum as well.

In the 4D spacetime, in contrast to the usual formulation of electromagnetism with the 3-vectors **E**, **B**, **j**, ... , there are no two laws, the Ampèr-Maxwell law and Gauss's law, but *only one law*, that is expressed by equation (16), i.e. equation (18), and the same for other two laws and equation (17).

The mathematical reason for such an inseparability is that, e.g., the gradient operator ∂ is a vector field defined on the 4D spacetime. If represented in some basis then its vector character remains unchanged only when *all its components together with associated basis vectors* are taken into account in the considered equation. The same holds for other vectors E, B, j, P, etc. and multivectors like F, $\mathcal{M}$, ... For example, in general, in the 4D spacetime, the current density vector j is a well-defined physical quantity, but it is not the case with the usual charge density ρ and the usual current density **j** as a 3-vector. Similarly, in general, the gradient operator ∂ cannot be divided into the usual time derivation and the spatial derivations. In the 4D spacetime, an independent physical reality is attributed to the position vector x, the gradient operator ∂, the current density vector j, the vectors of the electric and magnetic fields E and B, respectively, etc., but not to the 3-vector **r** and the time t, to the 3-vectors **j**, **E**, **B**, etc. Therefore, in the 4D spacetime, it is not possible to speak about the static case in the electromagnetism, i.e., about the electrostatics and magnetostatics.

An important consequence stems from the above mentioned inseparability of the 4D spacetime into the 3D space and the time and therefore from the inseparability of the equation (16), i.e. (18), into two laws, and similarly for the equation (17). It can be seen from Maxwell's equations with the 3-vectors, e.g. equations (30) and (31), and also (33) and (34), which all are given below, that *in the static case* the electric and magnetic fields, **E** and **B**, respectively, are completely decoupled. However, as already stated, in the 4D spacetime there is no static case. The equations (16), i.e. (18), and (17) reveal that *the vectors of the electric and magnetic fields E and B*, respectively, *are never decoupled.* This statement holds for the vacuum as well. Thus if, for example, we have a magnetization M (a permanent magnet) but without permanent polarization P and without $j^{(C)}$, then, as can be seen from (18), M will induce *both B and E.* Such a result is completely understandable because E and B are derived from *one* fundamental quantity, the electromagnetic field bivector F, by the decomposition of F (10) and by (11), and similarly P and M are derived from *one* quantity, the generalized magnetization-polarization bivector $\mathcal{M}$, by the decomposition of $\mathcal{M}$ (12) and by (13). The equations (5), i.e., its 'source representation' (7), are the basic field equations with the bivectors F and $\mathcal{M}$; F unites E and B and $\mathcal{M}$ unites P and M. Besides, F is independent on v and $\mathcal{M}$ is independent on u. The whole formulation of electromagnetism of moving media could be done exclusively in terms of F and $\mathcal{M}$ in the same way as in [2] for vacuum.

It is worth mentioning that in the integral form the equation that corre-

sponds to the local equation (16) can be obtained from the equation (21) in [2] replacing F by $F+\mathcal{M}/\varepsilon_0$, j by $j^{(C)}$ and inserting into it the decompositions (10) and (12), and similarly for (17) and the equation (18) in [2].

4. The basic field equations for moving media in the standard basis

Observe that equations (1) - (18) are all coordinate-free relations. If the abstract 4D geometric quantities from them are represented in some basis then their representations contain both, components *and basis vectors*. In order to compare (15) with the usual formulations of electromagnetism we have to represent all abstract quantities in (15) in the standard basis $\{\gamma_\mu\}$. Then, in the $\{\gamma_\mu\}$ basis, the second equation from (5), $\partial\cdot(\varepsilon_0 F+\mathcal{M})=j^{(C)}/c$, becomes

$$\partial_\alpha(\varepsilon_0 F^{\alpha\beta}+\mathcal{M}^{\alpha\beta})\gamma_\beta=c^{-1}j^{(C)\beta}\gamma_\beta, \tag{19}$$

or, in terms of E, B, P, M (the Ampèrian form), i.e., if equation (16) is written in the $\{\gamma_\mu\}$ basis, it becomes

$$\partial_\alpha\{\varepsilon_0[\delta^{\alpha\beta}{}_{\mu\nu}E^\mu v^\nu+c\varepsilon^{\alpha\beta\mu\nu}v_\mu B_\nu]+[\delta^{\alpha\beta}{}_{\mu\nu}P^\mu u^\nu+(1/c)\varepsilon^{\alpha\beta\mu\nu}M_\mu u_\nu]\}\gamma_\beta=j^{(C)\beta}\gamma_\beta, \tag{20}$$

where $\delta^{\alpha\beta}{}_{\mu\nu}=\delta^\alpha_\mu\delta^\beta_\nu-\delta^\alpha_\nu\delta^\beta_\mu$. Similarly, in the $\{\gamma_\mu\}$ basis, (17) becomes

$$\partial_\alpha(c\delta^{\alpha\beta}{}_{\mu\nu}B^\mu v^\nu+\varepsilon^{\alpha\beta\mu\nu}E_\mu v_\nu)\gamma_5\gamma_\beta=0. \tag{21}$$

Again, as for (17), equation (21) is the same as in vacuum. In equation (20), as in (15), i.e., (16), there are two different velocities u and v. The equation (20) does not appear in the entire previous literature. The equation (15), i.e., (16) and (17) and also (20) and (21) are the fundamental results that are obtained in this paper and they will enable an alternative, but viable, treatment of electromagnetism of moving media.

The equation (20) can be written in 'source representation' as

$$\begin{aligned}&\partial_\alpha\{\varepsilon_0[\delta^{\alpha\beta}{}_{\mu\nu}E^\mu v^\nu+c\varepsilon^{\alpha\beta\mu\nu}v_\mu B_\nu]\}\gamma_\beta\\ =\ &\{j^{(C)\beta}-\partial_\alpha[\delta^{\alpha\beta}{}_{\mu\nu}P^\mu u^\nu+(1/c)\varepsilon^{\alpha\beta\mu\nu}M_\mu u_\nu]\}\gamma_\beta,\end{aligned} \tag{22}$$

according to which the sources of the E and B fields are the true current density $j^{(C)}$ and the P and M vectors. Again, as for (18), it can be concluded that *it is not possible to separate the field equation with sources for the E field from that one for the B field.* As stated in the preceding section the Ampèr-Maxwell law and Gauss's law are *inseparably* connected in one law, the equation (20), i.e. equation (22). Also, Faraday's law and the law that expresses the absence of magnetic charge are *inseparably* connected in one law, the equation (21). The whole discussion presented at the end of section 3 applies in the same measure to (20), i.e. (22), and (21), but now the abstract quantities, e.g., $j^{(C)}$, ∂, E have

to be replaced by their representations in the standard basis, $j^{(C)} = j^{(C)\beta}\gamma_\beta$, $\partial = \gamma^\beta \partial_\beta$, $E = E^\mu \gamma_\mu$.

5. The difference between the UT and the LT

In this section, for the sake of further consideration, we briefly examine the fundamental difference between the UT of the 3-vectors and the LT of the 4D geometric quantities. First, it is worth mentioning an important result regarding the usual formulation of electromagnetism, as in [12-14] or [5], which is presented in [15] and discussed in [9] and [16]. It is argued in [15] that an individual vector has no dimension; the dimension is associated with the vector space and with the manifold where this vector is tangent. Hence, what is essential for the number of components of a vector field is the number of variables on which that vector field depends, i.e., *the dimension of its domain.* Thus, strictly speaking, the time-dependent $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$, $\mathbf{D}(\mathbf{r},t)$ etc. cannot be the 3-vectors, since they are defined on the spacetime. Therefore, we use the term 'vector' for a geometric quantity, which is defined on the spacetime and which always has in some basis of that spacetime, e.g., the standard basis $\{\gamma_\mu\}$, four components (some of them can be zero). Note that vectors are usually called the 4-vectors. However, an incorrect expression, the 3-vector, will still remain for the usual $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$, $\mathbf{D}(\mathbf{r},t)$ etc..

Moreover, recently, [17-21] and [9], it is proved that, contrary to the general belief, the UT of the 3-vectors of the electric and magnetic fields, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ respectively, see, e.g., equations (11.148) and (11.149) in Jackson's well-known textbook [5], differ from the LT (boosts) of the corresponding 4D quantities that represent the electric and magnetic fields. The usual transformations of $\mathbf{E}$ and $\mathbf{B}$ are first derived by Lorentz [22] and Poincaré [23], see also two fundamental Poincaré's papers with notes by Logunov [24], and independently by Einstein [4], and subsequently derived and quoted in almost every textbook and paper on relativistic electrodynamics. They are always considered to be the relativistically correct LT of $\mathbf{E}$ and $\mathbf{B}$. As explained in [17-21] and [9] the fundamental difference between the UT and the LT of the electric and magnetic fields is that in the UT, e.g., the components of the transformed $\mathbf{E}'$ are expressed by the mixture of components of $\mathbf{E}$ and $\mathbf{B}$, and similarly for $\mathbf{B}'$, equation (11.148) in [5] or, e.g., equations (10.119) in [25]. The UT for the 3-vector $\mathbf{E}$ (and similarly for $\mathbf{B}$) are given, e.g., by equations (11.149) in [5], or equations (18-40)-(18-43) in [13] and they are

$$\begin{aligned}
\mathbf{E}' &= \gamma(\mathbf{E} + \beta \times c\mathbf{B}) - (\gamma^2/(1+\gamma))\beta(\beta \cdot \mathbf{E}), \\
\mathbf{B}' &= \gamma(\mathbf{B} - (1/c)\beta \times \mathbf{E}) - (\gamma^2/(1+\gamma))\beta(\beta \cdot \mathbf{B}),
\end{aligned} \tag{23}$$

where $\mathbf{E}'$, $\mathbf{E}$, β and $\mathbf{B}'$, $\mathbf{B}$ are all 3-vectors. All what is stated for the 3-vectors $\mathbf{E}$ and $\mathbf{B}$ and their UT holds in the same measure for the couple of the 3-vectors $\mathbf{P}$ and $\mathbf{M}$ and their UT

$$\begin{aligned}\mathbf{P} &= \gamma(\mathbf{P}' + \beta \times \mathbf{M}'/c) - (\gamma^2/(1+\gamma))\beta(\beta \cdot \mathbf{P}'), \\ \mathbf{M} &= \gamma(\mathbf{M}' - \beta \times c\mathbf{P}') - (\gamma^2/(1+\gamma))\beta(\beta \cdot \mathbf{M}'), \end{aligned} \tag{24}$$

see the equations, e.g., (18-68) - (18-71) in [13], or (4.2) in [12], or (6.78a) and (6.81a) in [14], etc.

However, *the correct LT always transform the 4D algebraic object representing the electric field only to the electric field, and similarly for the magnetic field.*

This fundamental difference between the LT and the UT can be briefly explained using the results, e.g., from [9]. Let us introduce the frame of 'fiducial' observers as the frame in which the observers who measure fields E and B are at rest. That frame with the standard basis $\{\gamma_\mu\}$ in it is called the γ_0-frame. In the γ_0-frame $v = c\gamma_0$ and therefore E from (11) becomes $E = F \cdot \gamma_0$ and it transforms under the active LT in such a manner that both F and the velocity of the observer $v = c\gamma_0$ are transformed by the LT, see equation (6) in [9]. As explained in [9], *Minkowski, in section* 11.6 *in* [1]*, showed that both factors of the vector E, as the product of one bivector and one vector, has to be transformed by the LT.* However, it is worth mentioning that Minkowski in all other parts of [1] dealt with the usual 3-vectors $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, etc.. These correct LT give that

$$E' = E + \gamma(E \cdot \beta)\{\gamma_0 - (\gamma/(1+\gamma))\beta\}, \tag{25}$$

equation (13) in [9]. *In the same way vector B transforms and vectors P, M as well, but for P and M the LT, like* (25)*, are the transformations from the rest frame of the medium* ($u = c\gamma_0$). For boosts in the direction γ_1 one has to take in that equation that $\beta = \beta\gamma_1$ (on the l.h.s. is vector β and on the r.h.s. β is a scalar). Hence, in the standard basis and when $\beta = \beta\gamma_1$ that equation becomes

$$E'^\nu \gamma_\nu = -\beta\gamma E^1 \gamma_0 + \gamma E^1 \gamma_1 + E^2 \gamma_2 + E^3 \gamma_3, \tag{26}$$

what is equation (14) in [9]. The most important result is that *the electric field vector E transforms by the LT again to the electric field vector E'; there is no mixing with the magnetic field B.* (The same happens with P and M.)

The comparison with experiments in electromagnetism, the motional emf [18], the Faraday disk [19], and the Trouton-Noble experiment [2, 26], show that the approach with 4D geometric quantities and their LT always agrees with the principle of relativity and it is in a true agreement (independent of the chosen inertial reference frame and of the chosen system of coordinates in it) with all experiments in electromagnetism. Also, it is shown in the mentioned papers that such a true agreement does not exist in the usual approaches, e.g., [5], [12-14], [25], in which the electric and magnetic fields are represented by 3-vectors $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ that transform according to the UT (23). (The same conclusion about the true agreement between the approach with 4D geometric quantities and the well-known experiments that test special relativity is obtained in [27]. There, in [27], it is explicitly shown that the relativity of simultaneity,

the Lorentz contraction and the time dilation are not well-defined in the 4D spacetime. They are not the intrinsic relativistic effects, because they depend on the chosen synchronization.)

6. Observers are at rest in a stationary medium. Comparison with the usual formulation with the 3-vectors

Having briefly discussed the LT and the UT we go back to the discussion of equations (20) (i.e., (22)) and (21). In all relatively moving inertial frames of reference and for any system of coordinates in them every term in the considered equations is always the same, because all quantities are the Lorentz invariant quantities, e.g., $j^{(C)\beta}\gamma_\beta = j'^{(C)\beta}\gamma'_\beta = j_r^{(C)\beta}\gamma_{r,\beta} = ...$. The primed quantities are the Lorentz transforms of the unprimed ones in the $\{\gamma_\mu\}$ basis, whereas the last expression refers to the true current density in the $\{r_\mu\}$ basis with the 'r' synchronization, see, e.g. [28]. Observe that, in the 4D spacetime, *only if all components, together with the associated basis vectors,* are taken into account in every term then all terms are invariant under the passive LT and thus the whole equations (20) ((22)) and (21) remain unchanged for different relatively moving frames and for different systems of coordinates in them. Only in that case the physical quantities and the equations with them are correctly defined in the 4D spacetime and the principle of relativity is naturally satisfied. This means that, in general, it is not allowed to consider separately some parts of the 4D geometric quantities, or some parts of the equations with them, e.g., to take the part with γ_0 separately from those ones with γ_i in equations (20) ((22)) and (21). Thus, for example, in the 4D spacetime, only the whole current density $j^{(C)}$, the abstract vector from (16) and (18), or some of its representation, e.g., that one in the standard basis, $j = j^{(C)\beta}\gamma_\beta$, is well-defined physical quantity, but not the charge density, $j^{(C)0}$ component, or the spatial components $j^{(C)i}$ taken alone. From the viewpoint of the 4D geometric approach the physical meaning of the charge density ρ is not well-defined. It is the temporal component j^0/c for one observer, but it transforms by the LT to the temporal component and the spatial component as well for the relatively moving observer. The same holds for the gradient operator ∂ and its representation in the standard basis $\partial = \gamma^\beta\partial_\beta$ and for the other 4D geometric quantities.

This is particularly visible going to some nonstandard basis, like the $\{r_\mu\}$ basis, i.e. with the 'r' synchronization, see [28] and [16]. The unit vectors in the $\{\gamma_\mu\}$ basis and the $\{r_\mu\}$ basis are connected as $r_0 = \gamma_0$, $r_i = \gamma_0 + \gamma_i$. The components of any vector are connected in the same way as the components of the position vector x are connected, $x_r^0 = x^0 - x^1 - x^2 - x^3$, $x_r^i = x^i$, e.g. for the components of vector E it also holds that $E_r^0 = E^0 - E^1 - E^2 - E^3$, $E_r^i = E^i$. The inverse relations are $\gamma_0 = r_0$, $\gamma_i = r_i - r_0$ and, e.g., for the components of the current density vector j, $j^0 = j_r^0 + j_r^1 + j_r^2 + j_r^3$, $j^i = j_r^i$. Thus, even in the same frame, the charge density in the $\{\gamma_\mu\}$ basis ($j^0 = c\rho$) loses its usual meaning; it is expressed by the sum of all components in the $\{r_\mu\}$ basis. However, observe that, as already stated, $j = j^\mu\gamma_\mu = j_r^\mu r_\mu$ and the same

holds for $E = E^\mu \gamma_\mu = E_r^\mu r_\mu$, for B, for x, etc. This reveals that in the $\{r_\mu\}$ basis the space and time cannot be separated. Hence, in the 4D spacetime the usual interpretations of the physical quantities, e.g., the charge density ρ and the current density as a 3-vector $\mathbf{j}$, are not appropriate.

An independent physical reality can be attributed either to the abstract geometric quantities, e.g. vectors x, E, B, P, M, j, .. bivectors F, $\mathcal{M}$, .., or to their representations in different bases like $j^\mu \gamma_\mu$, $E_r^\mu r_\mu$, $M'^\beta \gamma'_\beta$, etc.

Let us consider equation (20) in the case when $u = v$, i.e., the observer frame is comoving with medium. In that case it can be taken that $v = u = c\gamma_0$ ($u^\mu = v^\mu = (c, 0, 0, 0)$), i.e., that the observers who measure fields are at rest in a stationary medium. Then, equation (20) becomes

$$\begin{aligned} \partial_\alpha \{\varepsilon_0[\delta^{\alpha\beta}{}_{\mu\nu} E^\mu (\gamma_0)^\nu + c\varepsilon^{\alpha\beta\mu\nu}(\gamma_0)_\mu B_\nu] &\quad + \\ [\delta^{\alpha\beta}{}_{\mu\nu} P^\mu (\gamma_0)^\nu + (1/c)\varepsilon^{\alpha\beta\mu\nu} M_\mu (\gamma_0)_\nu]\}\gamma_\beta &= c^{-1} j^{(C)\beta}\gamma_\beta, \end{aligned} \tag{27}$$

The equation (27) can be also written in the 'source representation' as

$$\begin{aligned} &\partial_\alpha \{\varepsilon_0[\delta^{\alpha\beta}{}_{\mu\nu} E^\mu (\gamma_0)^\nu + c\varepsilon^{\alpha\beta\mu\nu}(\gamma_0)_\mu B_\nu]\}\gamma_\beta \\ = \ &\{c^{-1} j^{(C)\beta} - \partial_\alpha [\delta^{\alpha\beta}{}_{\mu\nu} P^\mu (\gamma_0)^\nu + (1/c)\varepsilon^{\alpha\beta\mu\nu} M_\mu (\gamma_0)_\nu]\}\gamma_\beta. \end{aligned} \tag{28}$$

As already mentioned, the sources of both fields together, E and B, are the true current density $j^{(C)}$ and the polarization and magnetization vectors, P and M respectively.

The observer frame is the γ_0-frame, $v = c\gamma_0$, which, with (11), yields that $E^0 = B^0 = 0$ and $E^i = F^{i0}$, $B^i = (1/2c)\varepsilon^{ijk0} F_{jk}$. Furthermore, in the considered case, the γ_0-frame coincides with the rest frame of the medium. Hence, in that frame and with (13), it also holds that $P^0 = M^0 = 0$, $P^i = \mathcal{M}^{i0}$, $M^i = (c/2)\varepsilon^{0ijk}\mathcal{M}_{jk}$. Then, equation (27) becomes

$$\begin{aligned} [\partial_k (E^k + P^k/\varepsilon_0) - j^{(C)0}/c\varepsilon_0]\gamma_0 + &\quad \{c\varepsilon^{ijk0}\partial_j[(B_k - \mu_0 M_k] \\ -j^{(C)i}/c\varepsilon_0 - \partial_0(E^i + P^i/\varepsilon_0)\}\gamma_i &= 0 \end{aligned} \tag{29}$$

In vacuum, equation (29) coincides with the first two terms, i.e., the terms with γ_0 and γ_i, in equation (8) in [19]. In the approach with 4D geometric quantities it is not possible to make any further simplification. *In the 4D spacetime, only the whole equation* (29) *is physically meaningful and there is no physical sense in some parts of it, for example, to take the part with γ_0 separately from those ones with γ_i.* Note that in the approach with 3-vectors there is not any Maxwell's equation that corresponds to equation (29).

Let us write equation (29) as $a^\alpha \gamma_\alpha = 0$, in which, as can be easily recognized, the coefficients a^α correspond the usual Maxwell's equations in the component form. There are *two Maxwell equations* in the *component form*; the coefficient a^0 corresponds to the *component form* of the Gauss law for the electric field and the coefficients a^i correspond to the Ampère-Maxwell law in the *component form*. In [19], for the first time, a fundamental discovery is achieved that the usual Maxwell's equations with the 3-vectors (for vacuum) are not covariant

under the LT. In section 2.3 in [19], the active LT (equation (16) in [19]) are applied to the equation (8) in [19] (in vacuum, as already stated, our equation (29) corresponds to the first two terms of (8) from [19]). There, in that section, it is obtained that the coefficient a^0, which corresponds to the *component form* of the Gauss law for the electric field, does not transform by the LT again to the Gauss law but to a'^0, $a'^0 = \gamma a^0 - \beta\gamma a^1$, which is a combination of the Gauss law and a part of the Ampère-Maxwell law (a^1). (In our case, for the material medium, $a^0 = \partial_k(E^k + P^k/\varepsilon_0) - j^{(C)0}/c\varepsilon_0$.) If the Lorentz transformed equation (29), similarly as in equations (21)-(24) in [19], is expressed in terms of Lorentz transformed derivatives and Lorentz transformed vectors E, B, P, M and j, then we find the same equation for a'^0 as it is equation (24) in [19], $a'^0 = \{[\gamma(\partial'_k E'^k) - j'^0/c\varepsilon_0] + \beta\gamma[\partial'_1 E'^0 + c(\partial'_2 B'_3 - \partial'_3 B'_2)]\}$, but E'^α has to be replaced by $E'^\alpha + P'^\alpha/\varepsilon_0$, B'_k by $B'_k - \mu_0 M'_k$, and j'^0 by $j'^{(C)0}$. The same discussion holds here as it is presented after equation (24) in [19]. From that discussion, and the above mentioned replacements, one concludes that the LT do not transform the Gauss law into the 'primed' Gauss law but into quite different law; a'^0 contains the time component $E'^0 + P'^0/\varepsilon_0$, whereas $E^0 = P^0 = 0$, and also the new 'Gauss law' includes the derivatives of the magnetic field. The same situation happens with other Lorentz transformed terms, which again explicitly shows that *the Lorentz transformed Maxwell's equations are not of the same form as the original ones.* Hence, contrary to all previous considerations, and contrary to the general opinion, *the usual Maxwell's equations are not Lorentz covariant equations either in vacuum or in a material medium.* This result proves in another way that *in the 4D spacetime* only the whole equation (29) is physically meaningful and not its separate parts. Remember that equation (29) is derived from equation (27), i.e. from (20), for which it also holds that $a^\beta\gamma_\beta = a'^\beta\gamma'_\beta = a^\beta_r\gamma_{r,\beta} = ...$. Here, as before, the primed quantities are the Lorentz transforms of the unprimed ones in the $\{\gamma_\mu\}$ basis, whereas the last expression refers to the $\{r_\mu\}$ basis with the 'r' synchronization.

Let us see how the equation (29) could be compared with the usual form of Maxwell's equations for stationary media, which deals with the 3-vectors, e.g., [12-14], [25], [5], [10], etc. Obviously, the comparison will be possible *only* if the term with γ_0 is considered separately from those ones with γ_i and if in equation (29) *only the components are taken into account.* But, as explained above, *from the viewpoint of the geometric approach such a procedure is not correct in the 4D spacetime.*

If in equation (29), i.e. in $a^0\gamma_0 + a^i\gamma_i = 0$, one takes that $a^0 = 0$, and multiply the spatial components of E, P and $j^{(C)}$ from a^0 by *the unit 3-vectors* **i**, **j**, **k**, then the term with γ_0 will become the equation

$$\nabla \cdot \varepsilon_0 \mathbf{E}(\mathbf{r}, t) = \rho^{(C)}(\mathbf{r}, t) - \nabla \cdot \mathbf{P}(\mathbf{r}, t), \tag{30}$$

what is equation (9-6)(1) in [13], or equation (4.5) (3) in [12], etc. In the same way it will be obtained that the terms with γ_i will become the equation

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \mu_0[\mathbf{j}^{(C)}(\mathbf{r}, t) + \partial(\varepsilon_0\mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t))/\partial t + \nabla \times \mathbf{M}(\mathbf{r}, t)], \tag{31}$$

what is equation (9-6)(4) in [13], or equation (4.5) (2) in [12], or equation (4.114) in [14], etc. The way in which the equations with the 3-vectors (30) and (31) are constructed clearly shows that equation (29) is essentially different than equations (30) and (31).

Similarly, we find that in the γ_0-frame equation (21) becomes

$$(c^2\partial_k B^k)\gamma_5\gamma_0 - (c\partial_0 B^i + \varepsilon^{ijk0}\partial_j E_k)\gamma_5\gamma_i = 0. \tag{32}$$

The equation (32) coincides, without any changes, with the last two terms, i.e., the terms with $\gamma_5\gamma_0$ and $\gamma_5\gamma_i$, in the equation for vacuum, (8) in [19]. As already stated, in (32), Faraday's law and the law that expresses the absence of magnetic charge are *inseparably* connected in one law. But, as in the discussion of equation (29), we can make the comparison of equation (32) with the usual form of Maxwell's equations for stationary media, which deals with 3-vectors. The equation with 3-vectors that expresses the absence of magnetic charge

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \tag{33}$$

can be constructed from the term with $\gamma_5\gamma_0$ in (32) in the same way as equations (30) and (31) are constructed from equation (29). The obtained equation (33) is equation (9-6)(2) in [13], or equation (4.5) (4) in [12], etc. Similarly, the terms with $\gamma_5\gamma_i$ will give the equation with 3-vectors, Faraday's law,

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\partial\mathbf{B}(\mathbf{r}, t)/\partial t, \tag{34}$$

what is equation (9-6)(3) in [13], or equation (4.5) (1) in [12], etc. (The components of vectors E, B, P, M with superscripts (E^i, B^i, P^i, M^i) from (29) are identified with the components of the usual 3-vectors and $\varepsilon^{0123} = 1$.) As already mentioned, Maxwell's equations in terms of $\mathbf{E}$, $\mathbf{B}$, $\mathbf{P}$ and $\mathbf{M}$, equations (4.5) in [12], i.e. (30), (31), (33) and (34) here, are said to be in the Ampèrian form.

If, as usual, the electric displacement 3-vector $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ has been introduced together with the magnetic field intensity 3-vector $\mathbf{H} = (1/\mu_0)\mathbf{B} - \mathbf{M}$ then equations (30) and (31) become

$$\begin{aligned} \nabla \cdot \mathbf{D}(\mathbf{r}, t) &= \rho^{(C)}(\mathbf{r}, t), \\ \nabla \times \mathbf{H}(\mathbf{r}, t) &= \mathbf{j}^{(C)}(\mathbf{r}, t) + \partial\mathbf{D}/\partial t. \end{aligned} \tag{35}$$

In (35), the first equation is equation (9-7)(1) in [13], whereas the second one is equation (9-7)(4) in [13], or equation (4.116) in [14], etc.

According to this discussion there is an essential difference between the Maxwell equations (30), (31), or (35), with the 3-vectors and equation (29), i.e. the equations from which (29) is derived, (27), (20) and (16). In the 4D geometric approach there is *one law*, equation (29), i.e., (27), or (20), or (16), whereas in the approach with the 3-vectors there are *two laws*, equations (30) and (31), or (35). In order to obtain two laws (30), (31), or (35), from equation (29) we had to make several steps. First, the term with γ_0 is taken separately

from those ones with γ_i, then only the components in these terms are taken into account and finally the components are multiplied by the unit 3-vectors **i**, **j**, **k**. But, in the 4D spacetime, as explained above, these steps are not mathematically correct. This consideration clearly shows that equation (29), i.e., equations (27), or (20), or (16), from which (29) is derived, is not equivalent to equations (30), (31), or (35). The equation (29) is more general and, strictly speaking, it is not possible to obtain equations (30), (31), or (35) from (29) by a mathematically correct procedure in the 4D spacetime. The same consideration holds in the same measure for the relation between (32) and the equations with the 3-vectors (33) and (34).

Furthermore, in the usual approach with the 3-vectors one can speak about the static case. Then, equations (30) and (31), or the first and the second equation in (35), are completely decoupled, i.e., in the static case the electric and magnetic fields as the 3-vectors are decoupled. In the 4D geometric approach such a decoupling is never possible, because there is only one law in which there are both together E and B as vectors. The same consideration holds for the Maxwell equations (33) and (34) with the 3-vectors and equation (32), i.e. the equations from which (32) is derived, (21) and (17).

The most important difference is the following. The quantities entering into (29) and (32) are representations in the standard basis of the abstract 4D quantities from equation (15), i.e. equations (16) and (17). All these quantities are correctly defined in the 4D spacetime and they correctly transform under the LT (25) and (26), whereas it is not the case with quantities appearing in (30),(31), (33), (34) and (35), which transform according to UT (23), (24) and the corresponding UT of the 3-vectors **D**, **H**.

7. Observers are at rest in the laboratory frame, but material medium is moving. Comparison with the usual formulation with the 3-vectors

Now, let us examine equations (20) and (21) in the case of a moving material medium, but the observers are at rest in the laboratory frame, which will be denoted as the S frame. Then, in S, $v = c\gamma_0$, $v^\mu = (c, 0, 0, 0)$). Thus the laboratory frame is the γ_0-frame in which it holds that $E^0 = B^0 = 0$ and $E^i = F^{i0}$, $B^i = (1/2c)\varepsilon^{ijk0}F_{jk}$. Obviously, the equation (21) becomes the same as equation (32), i.e. the same as in vacuum and the whole discussion about the comparison of (32) with Maxwell's equations with 3-vectors remains unchanged. But, it is not so for equation (20).

In the γ_0-frame the considered medium is moving with velocity u, $u \neq c\gamma_0$, i.e., some of u^i are $\neq 0$. The rest frame of the medium will be denoted as the S' frame. For the sake of comparison with the usual formulation we present the considered equation in an expanded form in which the term with γ_0 and the terms with γ_i are explicitly written. Then, in the laboratory frame, which is

the γ_0-frame, equation (20) becomes

$$
\begin{aligned}
\{\partial_k[\varepsilon_0 E^k + c^{-1}(P^k u^0 - P^0 u^k) + c^{-2}\varepsilon^{kij0} M_i u_j] - c^{-1} j^{(C)0}\}\gamma_0 && + \\
\{-c^{-1} j^{(C)i} + (c\mu_0)^{-1}\varepsilon^{ijk0}\partial_j B_k - \varepsilon_0 \partial_0 E^i && + \\
c^{-1}\partial_\mu (P^\mu u^i - P^i u^\mu) - c^{-2}\varepsilon^{i\mu\alpha\beta}\partial_\mu M_\alpha u_\beta]\}\gamma_i &= 0 \quad (36)
\end{aligned}
$$

Again, as in the discussion of equation (29), it can be argued that *in the 4D spacetime, only the whole equation* (36) *is physically meaningful and there is no physical sense in some parts of it, for example, to take the part with* γ_0 *separately from those ones with* γ_i. Observe that in (36) there are terms with P^0 and M^0, which cannot exist in the usual formulation with the 3-vectors.

What will be obtained from (36) for the case of low velocities of the medium, i.e., for $\beta_u \ll 1$, $\gamma_u = (1-\beta_u^2)^{-1/2} \simeq 1$, where, in S and in the $\{\gamma_\mu\}$ basis, $u = u^\nu \gamma_\nu$, $u^\nu = (\gamma_u c, \gamma_u U^1, \gamma_u U^2, \gamma_u U^3)$, U^k are the components of the 3-velocity $\mathbf{U}$ and $\beta_u = |\mathbf{U}|/c$. To determine and compare P^0 and P^k in S *we use the LT of* P'^μ *from* S', the rest frame of the medium, and, for simplicity, it is taken that the medium, the S' frame, is moving along the common $+x^1$, x'^1 axis, i.e., $u^\nu = (\gamma_u c, \gamma_u U^1, 0, 0)$. In S', $P'^\mu = (0, P'^1, P'^2, P'^3)$. Then, as in equation (26), $P^\mu = (\beta_u \gamma_u P'^1, \gamma_u P'^1, P'^2, P'^3)$. Since $\beta_u \ll 1$ and $\gamma_u \simeq 1$ it follows that $P^0 \ll P^1$ and $P^k u^0 - P^0 u^k$ in (36) becomes $\simeq cP^k$, i.e., in that approximation $P^0 u^k$ can be neglected relative to $P^k u^0$. In the same way it can be concluded that $M^0 u^k$ can be neglected relative to $M^k u^0$. Therefore, with these approximations, equation (36) can be written as

$$
\begin{aligned}
\partial_k \varepsilon_0 E^k \gamma_0 + (c\mu_0)^{-1}\varepsilon^{ijk0}\partial_j B_k \gamma_i \simeq [c^{-1} j^{(C)0} - \partial_k P^k - \\
c^{-2}\varepsilon^{kij0}\partial_k M_i U_j]\gamma_0 + [c^{-1} j^{(C)i} + \partial_0(\varepsilon_0 E^i + P^i) \quad + \\
c^{-1}((U^k\partial_k)P^i - U^i(\partial_k P^k)) + c^{-2}\varepsilon^{ijk0}(\partial_0 M_j U_k + c\partial_j M_k]\gamma_i, \qquad (37)
\end{aligned}
$$

Notice that (37) *is obtained from* (36) *using the LT of the vectors* $P^\mu \gamma_\mu$ and $M^\mu \gamma_\mu$ *and not the UT of the 3-vectors* $\mathbf{P}$ and $\mathbf{M}$, (24). We see that $D^k = \varepsilon_0 E^k + P^k$ *and* $H^k = B^k/\mu_0 - M^k$ *can be introduced into equation* (37), *whereas such replacement is not possible for equation* (36), *due to the existence of the terms with* P^0 *and* M^0 *in* (36). (The part with γ_0 is $\partial_k D^k \gamma_0 = [c^{-1} j^{(C)0} - c^{-2}\varepsilon^{kij0}\partial_k M_i U_j]\gamma_0$, whereas the part with γ_i is $\varepsilon^{ijk0}\partial_j H_k \gamma_i = [j^{(C)i} + c\partial_0 D^i + ((U^k\partial_k)P^i - U^i(\partial_k P^k)) + c^{-1}\varepsilon^{ijk0}\partial_0 M_j U_k]\gamma_i$.)

The equation (37) could be compared with the usual form of Maxwell's equations for moving media, which deal with the 3-vectors, e.g., [12-14], [10], etc. Again, as in section 6, the comparison will be possible *only* if the term with γ_0 is considered separately from those ones with γ_i and if in equation (37) *only the components are taken into account.* As before, we argue that *from the viewpoint of the geometric approach such a procedure is not correct in the 4D spacetime.* If, as in section 6, in equation ((37), i.e. in $a^0\gamma_0 + a^i\gamma_i = 0$, one takes that $a^0 = 0$, and multiply the spatial components of E, P, M and $j^{(C)}$ from a^0 by *the unit 3-vectors* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ then the term with γ_0 will become the equation

$$
\nabla \cdot \varepsilon_0 \mathbf{E}(\mathbf{r},t) = \rho^{(C)}(\mathbf{r},t) - \nabla \cdot [\mathbf{P}(\mathbf{r},t) - c^{-2}(\mathbf{M}(\mathbf{r},t) \times \mathbf{U})]. \qquad (38)
$$

Usually, as, e.g., in [13] (the derivation of equations (9-18) (1-4)) the case of: 'a non-magnetized medium moving with a velocity $\mathbf{u}$ which is small compared with velocity of light,' is considered. This means that in (36) one has to take not only $\beta_u \ll 1$, which leads to (37), but also $M_i = 0$. Then, instead of the part with γ_0 from (37) one gets the equation $\partial_k D^k \gamma_0 = \rho^{(C)}\gamma_0$. In the formulation with 3-vectors that equation corresponds to, e.g., equation (9-18)(1) in [13], $\nabla \cdot \mathbf{D} = \rho^{(C)}$.

Similarly, the terms with γ_i from equation (37) could be compared with the usual form of Maxwell's equations for moving media, which deals with the 3-vectors, e.g., equation (9-18) (4) in [13], or the equations in Problem 6.8 in [14]. Then, the terms with γ_i from equation (37) correspond to the following equation with 3-vectors

$$\begin{aligned} \nabla \times \mathbf{B}(\mathbf{r}, t) \quad = \quad & \mu_0[\mathbf{j}^{(C)}(\mathbf{r}, t) + \partial(\varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t))/\partial t + \nabla \times (\mathbf{P}(\mathbf{r}, t) \times \mathbf{U}) \\ & + (1/c^2)\partial(\mathbf{U} \times \mathbf{M}(\mathbf{r}, t))/\partial t + \nabla \times \mathbf{M}(\mathbf{r}, t)]. \end{aligned} \tag{39}$$

The equation (39) can be written in the following form

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{j}^{(C)}(\mathbf{r}, t) + \partial \mathbf{D}(\mathbf{r}, t)/\partial t + (1/c^2)\partial(\mathbf{U} \times \mathbf{M}(\mathbf{r}, t))/\partial t + \nabla \times (\mathbf{P}(\mathbf{r}, t) \times \mathbf{U}), \tag{40}$$

where the 3-vectors $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ and $\mathbf{H} = (1/\mu_0)\mathbf{B} - \mathbf{M}$ have been introduced. Taking in equation (37) that not only $\beta_u \ll 1$ but that $M_i = 0$ as well, i.e., that a non-magnetized medium is considered, then instead of equation (40) we find

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \mu_0[\mathbf{j}^{(C)}(\mathbf{r}, t) + \partial \mathbf{D}(\mathbf{r}, t)/\partial t + \nabla \times (\mathbf{P}(\mathbf{r}, t) \times \mathbf{U})]. \tag{41}$$

This equation is the fourth equation in Problem 6.8 in [14]. It differs from equation (9-18) (4) in [13], which contains an additional term $\mu_0 \rho^{(C)} \mathbf{U}$. As seen from (41) the appearance of that additional term is not justified.

The whole consideration on the difference between Maxwell's equations with 3-vectors and the equations with the 4D quantities that is presented at the end of section 6 holds in the same measure here. However, the difference between these two approaches (3-vectors versus 4D geometric quantities) is even bigger for the case examined in this section. Namely, due to the existence of the terms with P^0 and M^0 in (36) that equation cannot be compared with Maxwell's equations with 3-vectors. The comparison can be made only for low velocities of the medium when equation (36) reduces to equation (37).

There is also an additional difference between Maxwell's equations with 3-vectors, e.g., equations (9-18) in [13], and our equations (36) and (37). It is stated in [13] (under equations (9-18)): 'Note that Maxwell's equations for moving (nonmagnetic) media in the form given by Eq. (9-18) (4) are 'mixed,' i.e., the sources $\mathbf{j}_{true}$, $\mathbf{P}$, ρ_{true}, are measured in the moving medium, while the fields are given in the stationary frame.' On the other hand, as already stated above, *all quantities* in (36) and (37) are determined in the laboratory frame, which is the γ_0-frame. Moreover, as already explained, all quantities in (36) and (37) are correctly defined in the 4D spacetime and they correctly transform under the LT, which is not the case with Maxwell's equations with the 3-vectors.

8. Discussion and Conclusions

There are several important differences between the field equations reported here and all others in the previous literature including the modern textbook on classical electrodynamics [29], which uses the calculus of exterior forms.

First, instead of dealing with the electromagnetic excitation $\mathcal{H}$ (8) and the field equation with it (9) we exclusively deal with the equations (5) for the electromagnetic field F and a matter field $\mathcal{M}$ as the primary equations for the electromagnetism in moving media. As discussed in sections 2 and 3, the expression for $\mathcal{H}$ (8) in terms of F and $\mathcal{M}$ is in some sense unsatisfactory, since F and $\mathcal{M}$ are physically different kind of entities. Furthermore, what is particularly important, in general, two different velocity vectors, v - the velocity of the observers and u - the velocity of the moving medium, enter into the decompositions of F and $\mathcal{M}$, the equations (10) and (12), respectively. For this reason we also do not deal with the decomposition of $\mathcal{H}$ (14) into the electric and magnetic excitations D and H, respectively, where $D = \varepsilon_0 E + P$ and $H = (1/\mu_0)B - M$. As stated in section 3, such a decomposition as (14) is possible if only one velocity, the velocity of the medium u, is taken into account, or the case $u = v$ is considered, or both decompositions (10) and (12) are made with the same velocity vector, either u or v. (Recently, the last case is considered in [30], but with F and $\mathcal{H}$.)

The second important difference refers to the interpretation of the field equations. The basic field equation (15) contains two different velocities u and v. It is also written as equations (16) and (17). From these equations it is visible that *in the 4D spacetime*, in contrast to the formulation of electromagnetism in terms of Maxwell's equations with the 3-vectors **E**, **B**, **P** and **M**, *there are no two laws, the Ampèr-Maxwell law and Gauss's law, but only one law, that is expressed by equation* (16), *i.e. equation* (18), *and the same for equation* (17) and *Faraday's law and the law that expresses the absence of magnetic charge.*

Furthermore, the interesting results are obtained in sections 6 and 7. There, the field equations, written in the standard basis (20), i.e. (22), and (21), are compared with the usual form (with the 3-vectors) of Maxwell's equations for moving media. In section 6, it is shown that the comparison is possible *only* if the term with γ_0 is considered *separately* from those ones with γ_i and if in equations (29) and (32) *only the components* are taken into account. In order to get the usual equations with the 3-vectors, (30),(31), (33), (34) and (35), these components have to be multiplied by *the unit 3-vectors* **i**, **j**, **k**. Moreover, as shown in section 7, such a procedure is not applicable to equation (36), but only to equation (37), which is derived from (36) for the case of low velocities of the medium. As explained at the end of section 6, *the above mentioned steps in the comparison are not mathematically correct in the 4D spacetime.* Hence, in the 4D spacetime, the equations, e.g. (15), (16), (17), (20), (21), ... , with 4D geometric quantities E, B, P and M that correctly transform under the LT (25) and (26), *are not equivalent* to the usual Maxwell equations, e.g.,(30), (31), (33), (34), (35), (38), (39), ... , with the 3-vectors **E**, **B**, **P** and **M** that transform according to UT (23), (24).

The theoretical and experimental consequences of the results that are obtained in this paper will be carefully examined. An interesting consequence that can be experimentally examined is already mentioned in connection with equation (18). There, it is stated that if we have a magnetization M, a permanent magnet, moving *or stationary*, but without permanent polarization P and without $j^{(C)}$, then, as can be seen from (18), M will induce *both* B *and* E. An additional support to such an interpretation of equation (18) comes from the result that both B and E are vectors that correctly transform under the LT (25) and (26), which means that under the LT the vector E transforms again to the electric field E'. Hence, according to the LT (25) and (26), if there is an electric field outside *moving* magnet *it would necessary need to exist outside the same but stationary magnet*.

References

[1] Minkowski H 1908 *Nachr. Ges. Wiss. Göttingen* 53
Minkowski H 1910 *Math. Ann.* **68** 472
Saha M N and Bose S N 1920 *The Principle of Relativity: Original Papers by A. Einstein and H. Minkowski* (Calcutta: Calcutta University Press) (Engl. Trans.)

[2] Ivezić T 2005 *Found. Phys. Lett.* **18** 401

[3] Hestenes D 1966 *Space-Time Algebra (*New York: Gordon & Breach)
Hestenes D and Sobczyk G 1984 *Clifford Algebra to Geometric Calculus* (Dordrecht: Reidel)
Doran C. and Lasenby A 2003 *Geometric algebra for physicists* (Cambridge: Cambridge University Press)

[4] Einstein A 1905 *Ann. Physik.* **17** 891 translated by Perrett W and Jeffery G B 1952 in *The Principle of Relativity* (New York: Dover)

[5] Jackson J D 1998 *Classical Electrodynamics* 3rd edn (New York: Wiley)

[6] Hehl F W 2008 *Annalen der Physik* **17** 691

[7] Y N Obukhov 2008 *Annalen der Physik* **17** 830

[8] Y Itin and Y Friedman 2008 *Annalen der Physik* **17** 769

[9] Ivezić T 2010 *Phys. Scr.* **82** 055007

[10] Møller C 1972 *The Theory of Relativity* 2nd edn (Oxford: Clarendon Press)

[11] Hillion P 1993 *Phys. Rev. E* **48** 3060
Dereli T, Gratus J and Tucker R W 2007 *Phys. Lett. A* **361** 190

[12] Bladel J Van 1984 *Relativity and Engineering* (Berlin: Springer-Verlag)

[13] Panofsky W K H and Phillips M 1962 *Classical electricity and magnetism* 2nd edn (Reading: Addison-Wesley)

[14] Rosser W G W 1968 *Classical Electromagnetism via Relativity* (New York: Plenum)

[15] Oziewicz Z 2008 *Rev. Bull. Calcutta Math. Soc.* **16** 49
Oziewicz Z and Whitney C K 2008 *Proc. Nat. Phil. Alliance (NPA)* **5** 183 (also at http://www.worldnpa.org/php/)
Oziewicz Z 2009 Unpublished results that can be obtained

from the author at oziewicz@unam.mx
[16] Ivezić T 2010 *Phys. Scr.* **81** 025001
[17] Ivezić T 2003 *Found. Phys.* **33** 1339
[18] Ivezić T 2005 *Found. Phys.* Lett. **18** 301
[19] Ivezić T 2005 *Found. Phys.* **35** 1585
[20] Ivezić T 2008 *Fizika A* **17** 1
[21] Ivezić T 2008 *arXiv:* 0809.5277
[22] Lorentz H A 1904 *Proceedings of the Royal Netherlands Academy of Arts and Sciences* **6** 809
[23] Poincaré H 1906 *Rend. del Circ. Mat. di Palermo* **21** 129
[24] Logunov A A 1996 *Hadronic J.* **19** 109
[25] Griffiths D J 1989 *Introduction to Electrodynamics* 2nd edn (Englewood Cliffs: Prentice-Hall)
[26] Ivezić T 2007 *Found. Phys.* **37** 747
[27] Ivezić T 2002 *Found. Phys. Lett.* **15** 27
Ivezić T 2001 arXiv: *physics*/0103026
Ivezić T 2001 arXiv: *physics*/0101091
[28] Ivezić T 2001 *Found. Phys.* **31** 1139
[29] Hehl F W and Obukhov Yu N 2003 *Foundations of Classical Electrodynamics: Charge, flux, and metric* (Boston: Birkhäuser)
[30] Goto Shin-itiro, Tucker R W and Walton T J 2010 arXiv: 1003.1637